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Unicity Subspaces in L^1-Approximation*

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1. INTRODUCTION

In 1918 A. Haar [4] characterized those n-dimensional subspaces U_n of $C[0, 1]$ for which there exists a unique best approximation to every real-valued function $f \in C[0, 1]$ in the uniform (L^∞) norm. Haar proved that uniqueness always holds if and only if U_n is a Tchebycheff (T-) system on $[0, 1]$, or what is sometimes referred to as a Haar system (or Haar space). This simply means that no $u \in U_n \backslash \{0\}$ has more than $n-1$ zeros in $[0, 1]$. This is an intrinsic condition on U_n and is generally easily checked.

It is natural to consider this same problem in the $L^1[0, 1]$ norm. A first result was obtained by Jackson [6], who proved that if $U_n = \pi_{n-1}$, the space of algebraic polynomials of degree at most $n-1$, then there exists a unique best approximant to each $f \in C[0, 1]$. Two more general results were proved by Krein [8] in 1938. He showed that given any U_n, as above, there exist $f \in L^1[0, 1]$ with more than one best approximant. (This result was later reproved by Moroney [11].) He also generalized Jackson's result by proving that if U_n is a T-system on $(0, 1)$ then uniqueness holds for every $f \in C[0, 1]$. However, unlike the situation in the uniform norm, it is not necessary that U_n be a T-system in order that uniqueness hold. Thus the search for intrinsic, easily verified, necessary, and sufficient conditions on U_n ensuring uniqueness to every $f \in C[0, 1]$ in the L^1-norm has continued. Results in this direction were obtained by Ptak [13], Kripke and Rivlin [9], and Singer [15]. The first complete solution to this problem seems to be due to Cheney and Wulbert [2]. Unfortunately the conditions they set forth are not at all easily verifiable for specific U_n. More recently, Strauss [20] has given conditions which are somewhat different, but which are still not easily checked. The fact that nothing comparable in simplicity to Haar's theorem has been obtained is not surprising. The problem does

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not lend itself to such a solution. One of the reasons for this is that Haar's theorem remains valid when we alter the norm to any L^∞ norm with positive, continuous weight function $w(x)$, i.e.,

$$\|f\|_{\infty,w} = \max\{|f(x)|\, w(x): 0 \leqslant x \leqslant 1\}.$$

On the other hand it is easily seen that the criteria for the L^1 problem is weight function dependent. Thus, for example, in the simplest case $n=1$, the subspace $U_1 = \text{span}\{x+c\}$ for fixed $c \in \mathbb{R}$ provides a unique best approximation to every $f \in C[0, 1]$ if and only if $c \neq -\frac{1}{2}$. The presence of a weight function, however, will give rise to different $c \in \mathbb{R}$ for which uniqueness does not hold.

On the basis of work of Strauss [21] a condition (now termed the A-property) was formulated which was sufficient to guarantee uniqueness. This condition is in many instances verifiable and it made possible the amalgamation of various disparate results which had been obtained for specific subspaces. For example, Galkin [3] showed that for splines with fixed knots uniqueness always holds and Carroll and Braess [1] proved uniqueness for a space of continuous functions obtained by pasting together T-systems. A more general class of spaces, including those mentioned above, was considered by Sommer [16, 17], who used the A-property to obtain his results.

In this paper we study the problem of characterizing all those subspaces U_n for which there is uniqueness of the best approximant from U_n to each $f \in C[0, 1]$ in every L^1_w-norm, where w is a positive continuous weight function and

$$\|f\|_w = \int_0^1 |f(x)|\, w(x)\, dx.$$

This problem was first considered by S. J. Havinson [5] in 1958. He proved that if U_n has the property that no $u \in U_n \backslash \{0\}$ vanishes on an interval, and if uniqueness holds to each $f \in C[0, 1]$ in every L^1_w-norm for all $w \in \tilde{W}$, where $\tilde{W}$ is the set of measurable, bounded functions on $[0, 1]$ for which $\inf\{w(x): x \in [0, 1]\} > 0$, then U_n is necessarily a T-system on $(0, 1)$.

In this paper we prove two main results. Firstly, we prove that under minor restrictions on U_n, the A-property, alluded to earlier, is equivalent to uniqueness of the best approximant from U_n to each $f \in C[0, 1]$ in every L^1_w-norm for all strictly positive $w \in C[0, 1]$. That is, we restrict our class of weight functions w to

$$W = \{w: w \in C[0, 1], w(x) > 0, x \in [0, 1]\}.$$

Secondly, without any restriction on the U_n, we explicitly characterize all U_n which satisfy the A-property. This equivalent characterization should be compared with the sufficient condition given by Sommer in [16, 17]. The A-property implies that U_n is a very spline-like space.

After these results were obtained, Professor Strauss kindly sent the author two new manuscripts on this problem. In the first of these, Sommer [18] proved that if U_n satisfies the A-property then U_n is necessarily a weak Tchebycheff (WT-) system. The other paper by Kroó [10] proves that U_n satisfies the A-property if and only if uniqueness of the best approximant holds for each $f \in C[0, 1]$ in every L^1_w-norm where $w \in \tilde{W}$. It should be noted that we prove this latter result for a more restrictive set of weight functions (the "only if" part in both cases is an immediate consequence of Strauss [20]). However, we must pay a price in that we somewhat limit the permissible U_n.

2. Uniqueness in the L^1_w-Norm, w Fixed

Let $w \in C[0, 1]$ be a *fixed*, strictly positive function. We first review the known results of Cheney and Wulbert [2] and Strauss [21] on the question of characterizing those n-dimensional subspaces U_n of $C[0, 1]$ for which there exists a unique best approximant to each $f \in C[0, 1]$ in the L^1_w-norm, i.e.,

$$\|f\|_w = \int_0^1 |f(x)|\, w(x)\, dx.$$

For ease of exposition we shall say that U_n is a *unicity space* with respect to w if there exists a unique best approximant from U_n to each $f \in C[0, 1]$ in the L^1_w-norm.

We first prove, for completeness, the well-known characterization of best approximants. Let us set, for $f \in C[0, 1]$,

$$Z(f) = \{x : f(x) = 0\}$$

and

$$N(f) = [0, 1] \backslash Z(f).$$

THEOREM 2.1. *$u^* \in U_n$ is a best approximant to $f \in C[0, 1]$ in the L^1_w-norm if and only if*

$$\left| \int_0^1 u(x) \operatorname{sgn}(f(x) - u^*(x))\, w(x)\, dx \right| \leqslant \int_{Z(f-u^*)} |u(x)|\, w(x)\, dx \quad (1.1)$$

for all $u \in U_n$.

Proof. Assume (1.1) holds. Then for each $u \in U_n$

$$\begin{aligned}
\|f-u^*\|_w &= \int_{N(f-u^*)} (f-u^*)(x)\,\mathrm{sgn}((f-u^*)(x))\,w(x)\,dx \\
&= \int_{N(f-u^*)} (f-u)(x)\,\mathrm{sgn}((f-u^*)(x))\,w(x)\,dx \\
&\quad + \int_{N(f-u^*)} (u-u^*)(x)\,\mathrm{sgn}((f-u^*)(x))\,w(x)\,dx \\
&\leqslant \int_{N(f-u^*)} |(f-u)(x)|\,w(x)\,dx \\
&\quad + \int_{Z(f-u^*)} |(u-u^*)(x)|\,w(x)\,dx \\
&= \int_{N(f-u^*)} |(f-u)(x)|\,w(x)\,dx \\
&\quad + \int_{Z(f-u^*)} |(u-f)(x)|\,w(x)\,dx \\
&= \|f-u\|_w.
\end{aligned}$$

Now, let u^* be a best approximant to f. By an application of the Hahn–Banach Theorem, there exists a $g \in L^\infty[0,1]$ for which

(i) $\|g\|_\infty = 1$,

(ii) $(g, u)_w = 0$, for all $u \in U_n$,

(iii) $(g, f-u^*)_w = \|f-u^*\|_w$,

where $(g, h)_w = \int_0^1 g(x)\,h(x)\,w(x)\,dx$.

From (i) and (iii), it follows that $g(x) = \mathrm{sgn}((f-u^*)(x))$ a.e. on $N(f-u^*)$. From (ii), for each $u \in U_n$,

$$\int_{N(f-u^*)} g(x)\,u(x)\,w(x)\,dx = -\int_{Z(f-u^*)} g(x)\,u(x)\,w(x)\,dx.$$

Thus

$$\begin{aligned}
&\left|\int_{N(f-u^*)} u(x)\,\mathrm{sgn}((f-u^*)(x))\,w(x)\,dx\right| \\
&\quad = \left|\int_{Z(f-u^*)} g(x)\,u(x)\,w(x)\,dx\right| \\
&\quad \leqslant \int_{Z(f-u^*)} |u(x)|\,w(x)\,dx
\end{aligned}$$

by (i). ∎

In 1969, Cheney and Wulbert [2] characterized the unicity spaces U_n with respect to w. The characterization will be less useful to us than the proof thereof.

THEOREM 2.2 (Cheney and Wulbert [2]). *Let U_n be an n-dimensional subspace of $C[0, 1]$. Then U_n is a unicity space with respect to w if and only if there does not exist a measurable set A of $[0, 1]$ with boundary Y such that*

(i) $\int_A u(x)\, w(x)\, dx = \int_{[0,1]\setminus A} u(x)\, w(x)\, dx$, *for all* $u \in U_n$,

(ii) *there exists a* $u^* \in U_n \setminus \{0\}$ *such that* $u^*(x) = 0$ *for all* $x \in Y \cap (0, 1)$.

Proof. Assume that there exists an A and u^* satisfying (i) and (ii). We construct an $f \in C[0, 1]$ with more than one best approximant. To do so, set

$$h(x) = \begin{cases} 1, & x \in A, \\ -1, & x \in [0, 1]\setminus A. \end{cases}$$

From (i) it follows that

$$\int_0^1 u(x)\, h(x)\, w(x)\, dx = 0$$

for all $u \in U_n$. Let $f(x) = |u^*(x)|\, h(x)$. Since $u^*(x) = 0$ for all $x \in Y \cap (0, 1)$, it follows that $f \in C(0, 1)$. Redefine h at 0 and 1 so that $f \in C[0, 1]$. This is possible and we still maintain the above orthogonality. Now, for any choice of $u \in U_n$,

$$\begin{aligned} \|f - u\|_w &= \int_0^1 |f(x) - u(x)|\, w(x)\, dx \\ &\geqslant \int_0^1 h(x)(f(x) - u(x))\, w(x)\, dx \\ &= \int_0^1 h(x)\, f(x)\, w(x)\, dx \\ &= \int_0^1 |u^*(x)|\, w(x)\, dx \\ &= \|u^*\|_w \\ &= \|f\|_w. \end{aligned}$$

Thus $\min\{\|f-u\|_w: u\in U_n\}=\|f\|_w$. Furthermore, for $|\lambda|<1$, $\operatorname{sgn}(f(x)-\lambda u^*(x))=\operatorname{sgn}(h(x)\,|u^*(x)|-\lambda u^*(x))=\operatorname{sgn}(h(x)\,|u^*(x)|)$. Thus

$$\begin{aligned}\|f-\lambda u^*\|_w &= \int_0^1 |f(x)-\lambda u^*(x)|\, w(x)\, dx\\ &= \int_0^1 (f(x)-\lambda u^*(x))\operatorname{sgn}(h(x)\,|u^*(x)|)\, w(x)\, dx\\ &= \int_0^1 |u^*(x)|\, w(x)\, dx\\ &= \|f\|_w.\end{aligned}$$

It follows that λu^* is a best approximant to f for all λ, $|\lambda|<1$.

To prove the converse, one assumes the existence of an $f\in C[0,1]$ with at least two best approximants u_1 and u_2 from U_n, $u_1\not\equiv u_2$. Set $u^*=u_1-u_2$ and $\tilde{u}=(u_1+u_2)/2$. Since the set of best approximants is convex, $\tilde{u}$ is also a best approximant to f. It now easily follows that

$$|(f-\tilde{u})(x)|=|(f-u_1)(x)|/2+|(f-u_2)(x)|/2 \qquad \text{for all } x\in[0,1].$$

Thus if $x\in Z(f-\tilde{u})$, i.e., $(f-\tilde{u})(x)=0$, then $(f-u_1)(x)=(f-u_2)(x)=0$, which implies that $u^*(x)=0$. That is, $Z(f-\tilde{u})\subseteq Z(u^*)$.

Since $\tilde{u}$ is a best approximant to f it follows from the Hahn–Banach Theorem that there exists a $g\in L^\infty$ for which

(i) $\|g\|_\infty=1$,
(ii) $(g,u)_w=0$ for all $u\in U_n$,
(iii) $(g,f-\tilde{u})_w=\|f-\tilde{u}\|_w$.

Furthermore, as in Theorem 2.1, and from (i) and (iii), it follows that $g(x)=\operatorname{sgn}((f-\tilde{u})(x))$ a.e. on $N(f-\tilde{u})$. We assume that $g(x)=\operatorname{sgn}((f-\tilde{u})(x))$ for all $x\in N(f-\tilde{u})$. By a lemma of Phelps [12] (a simple application of Liapunov's theorem) it may be shown that one may choose g as above with $|g(x)|=1$ for all x.

Set $A=\{x: g(x)=1\}$. Then from (ii)

$$\int_A u(x)\,w(x)\,dx=\int_{[0,1]\setminus A} u(x)\,w(x)\,dx$$

for all $u\in U_n$. Y, the boundary of A, is such that each $x\in Y\cap(0,1)$ is a discontinuity point of g. These, by construction, are contained in the closed set $Z(f-\tilde{u})$, which is itself contained in $Z(u^*)$. Thus on $Y\cap(0,1)$, $u^*(x)=0$. This completes the proof. ∎

The next result was proved by Strauss [20, 21] in 1977. The original proof did not use Theorem 2.2.

Let $U_n^* = \{g\colon g \in C[0,1],\ |g(x)| = |u(x)| \text{ for some } u \in U_n\}$. Obviously $U_n \subseteq U_n^*$. U_n^* is generally substantially larger than U_n and need not be a subspace of $C[0, 1]$.

THEOREM 2.3 (Strauss [20, 21]). *U_n is a unicity space with respect to w if and only if the zero function is not a best approximant to any $g \in U_n^* \setminus \{0\}$ from U_n in the L_w^1-norm.*

Proof. Assume U_n is not a unicity space. Let A, Y and $u^* \in U_n \setminus \{0\}$ be as in Theorem 2.2, and set

$$h(x) = \begin{cases} 1, & x \in A, \\ -1, & x \in [0,1] \setminus A. \end{cases}$$

It was shown, in the proof of Theorem 2.2, that $f^*(x) = h(x)\,|u^*(x)| \in C[0, 1]$ has more than one best approximant, one of which is the zero function. Furthermore, $|f^*(x)| = |u^*(x)|$ so that $f^* \in U_n^* \setminus \{0\}$.

Assume now that $g \in U_n^* \setminus \{0\}$ and the zero function is a best approximant to g from U_n. Let $u^* \in U_n \setminus \{0\}$ be such that $|g(x)| = |u^*(x)|$ for all x. Since the zero function is a best approximant to g, it follows from Theorem 2.1 that

$$\left| \int_0^1 u(x) \operatorname{sgn}(g(x))\, w(x)\, dx \right| \leqslant \int_{Z(g)} |u(x)|\, w(x)\, dx \qquad \text{for all } u \in U_n.$$

For λ, $|\lambda| < 1$, consider $g(x) - \lambda u^*(x)$. Since $|g(x)| = |u^*(x)|$, it is easily seen that $Z(g) = Z(g - \lambda u^*)$ and $\operatorname{sgn}(g(x)) = \operatorname{sgn}(g(x) - \lambda u^*(x))$ for all x. Thus

$$\left| \int_0^1 u(x) \operatorname{sgn}(g(x) - \lambda u^*(x))\, w(x)\, dx \right| \leqslant \int_{Z(g - \lambda u^*)} |u(x)|\, w(x)\, dx$$

for all $u \in U_n$, implying that λu^* is also a best approximant to g for all λ, $|\lambda| < 1$. ∎

An equivalent form of Theorem 2.3 is given by the following statement, based on the characterization of Theorem 2.1.

COROLLARY 2.4 (Strauss [21]). *U_n is a unicity space with respect to w if and only if to each $g \in U_n^* \setminus \{0\}$ there exists a $u_g \in U_n$ for which*

$$\int_0^1 u_g(x) \operatorname{sgn}(g(x))\, w(x)\, dx > \int_{Z(g)} |u_g(x)|\, w(x)\, dx.$$

On the basis of Corollary 2.4 we may now define the A-property introduced by Strauss (see [21]).

DEFINITION 2.1. We say that the n-dimensional subspace U_n of $C[0, 1]$ satisfies the A-*property* if for every $g \in U_n^* \backslash \{0\}$ there exists a $u \in U_n \backslash \{0\}$ for which

(i) $u(x) = 0$ a.e. on $Z(g)$,

(ii) $u(x) \operatorname{sgn}(g(x)) = |u(x)|$ for all $x \in [0, 1] \backslash Z(g)$.

Note that this definition is independent of w. A totally equivalent formulation of the A-property is the following. We record it because it is this formulation which we use.

DEFINITION 2.1'. Let $u^* \in U_n \backslash \{0\}$. Let $I_i = (a_i, b_i)$, $i = 1,..., m$ (m may be infinite, but is countable), denote the maximum open intervals of $(0, 1)$ on which u^* does not vanish. That is, $u^*(x) \neq 0$, $x \in I_i$, and $u^*(x) = 0$ for all $x \in (0, 1) \backslash \bigcup_{i=1}^m I_i$. Let $\boldsymbol{\varepsilon} = (\varepsilon_1,..., \varepsilon_m)$, $\varepsilon_i \in \{-1, 1\}$, $i = 1,..., m$. Then U_n satisfies the A-*property* if for each $u^* \in U_n \backslash \{0\}$ and for every choice of $\boldsymbol{\varepsilon}$, as above, there exists a $u_{\boldsymbol{\varepsilon}} \in U_n \backslash \{0\}$ for which

(i) $u_{\boldsymbol{\varepsilon}}(x) = 0$ a.e. on $Z(u^*)$,

(ii) $\varepsilon_i u_{\boldsymbol{\varepsilon}}(x) \geqslant 0$, $x \in I_i$, $i = 1,..., m$.

Let $W = \{w: w \in C[0, 1], w(x) > 0 \text{ for all } x \in [0, 1]\}$. From Corollary 2.4 we have

THEOREM 2.5 (Strauss [21]). *Assume U_n satisfies the* A-*property. Then U_n is a unicity space with respect to each $w \in W$.*

Proof. Let $g \in U_n^* \backslash \{0\}$ and let $u \in U_n \backslash \{0\}$ be as in Definition 2.1. From (i),

$$\int_{Z(g)} |u(x)| \, w(x) \, dx = 0.$$

From (ii),

$$\int_0^1 u(x) \operatorname{sgn}(g(x)) \, w(x) \, dx = \int_0^1 |u(x)| \, w(x) \, dx > 0.$$

Thus uniqueness follows from Corollary 2.4. ∎

Remark. Based on Theorems 2.2 or 2.3, or the A-property, it is easily shown that if U_n is a T-system on $(0, 1)$, then U_n is a unicity space for all $w \in W$. One may also use the above results to prove uniqueness for various other subspaces, see, e.g., Sommer [16, 17].

3. Unicity Spaces and the A-Property

The previous result proved that the A-property for U_n is sufficient to ensure that U_n is a unicity space in the L^1_w-norm for every $w \in W$. Under minor assumptions on U_n, we prove the converse.

Let A be a measurable subset of $[0, 1]$. For notational ease, let $|A|$ denote its Lebesgue measure.

Theorem 3.1. *Let U_n be an n-dimensional subspace of $C[0, 1]$. Assume that for every $u \in U_n$, $|Z(u)| = |\text{int}(Z(u))|$. If U_n is a unicity space in the L^1_w-norm for every $w \in W$, then U_n satisfies the* A-*property.*

Remark. We recall that without any restriction on U_n, this result was recently proved by Kroó [10] if we replace W by the set of all bounded, measurable, strictly positive functions.

Proof. Assume that U_n does not satisfy the A-property. By Definition 2.1′ there then exists a $u^* \in U_n \backslash \{0\}$ such that the following hold:

Let I_i, $i = 1,..., m$, be maximal non-zero open intervals of u^* in $(0, 1)$ (m may be infinite). There exists $\varepsilon^* = (\varepsilon_1^*,..., \varepsilon_m^*)$, $\varepsilon_i^* \in \{-1, 1\}$, $i = 1,..., m$, such that no $u \in U_n \backslash \{0\}$ satisfies

(i) $u(x) = 0$ a.e. on $Z(u^*)$,

(ii) $\varepsilon_i^* u(x) \geqslant 0$, $x \in I_i$, $i = 1,..., m$.

Let $\tilde{U} = \{u: u \in U_n, u(x) = 0 \text{ a.e. on } Z(u^*)\}$. Obviously $\tilde{U}$ is a subspace of U_n of dimension k, $1 \leqslant k \leqslant n$ ($k \geqslant 1$ since $u^* \in \tilde{U}$). Let $\tilde{U} = \text{span}\{u_1,..., u_k\}$. Define $\tilde{V} = \text{span}\{v_1,..., v_k\}$ where

$$v_i(x) = \tilde{h}(x)\, u_i(x), \qquad i = 1,..., k,$$

and

$$\tilde{h}(x) = \begin{cases} \varepsilon_i^*, & x \in I_i,\ i = 1,..., m, \\ 0, & \text{otherwise.} \end{cases}$$

From the above it follows that there exists no $v \in \tilde{V} \backslash \{0\}$ for which $v(x) \geqslant 0$ for almost all x.

We claim that there exists a $\tilde{w} \in W$ for which

$$\int_0^1 u(x)\, \tilde{h}(x)\, \tilde{w}(x)\, dx = 0, \qquad \text{for all } u \in \tilde{U}.$$

Consider $V = \{v(x)\, dx: v \in \tilde{V}\}$ as a subset of $M[0, 1]$ ($= C'[0, 1]$), the set of real Borel measures of bounded total variation on $[0, 1]$, i.e., the dual space of $C[0, 1]$. We shall consider $M[0, 1]$ endowed with the weak*-topology induced by $C[0, 1]$. Set

$$K = \left\{\mu\colon d\mu \in M[0, 1], d\mu \geqslant 0, \int_0^1 d\mu = 1\right\},$$

i.e., K is the set of probability densities on $[0, 1]$.

In the weak*-topology on $M[0, 1]$, K is a compact, convex subset of $M[0, 1]$, while $\tilde{V}$ is a finite-dimensional subspace. Furthermore, $K \cap \tilde{V} = \emptyset$ since no $v \in \tilde{V}$ is non-negative a.e. on $[0, 1]$. As such, there exists (see, e.g., Schaefer [14]) a continuous linear functional $\tilde{w}$ on $M[0, 1]$ such that $\tilde{w}(v) = 0$ for all $v \in \tilde{V}$ and $\tilde{w}(\mu) > 0$ for all $\mu \in K$. In the weak*-topology, continuous linear functionals may be represented by functions in $C[0, 1]$. Without abusing notation we also denote this representing function by $\tilde{w}$. Thus $\tilde{w} \in C[0, 1]$, and

$$\int_0^1 v(x)\, \tilde{w}(x)\, dx = 0, \qquad \text{all } v \in \tilde{V},$$

$$\int_0^1 \tilde{w}(x)\, d\mu(x) > 0, \qquad \text{all } \mu \in K.$$

Since δ_x (the point functional at x) is in K for each $x \in [0, 1]$, it follows that $\tilde{w}(x) > 0$ for all $x \in [0, 1]$. Thus $\tilde{w} \in W$, and

$$\int_0^1 u(x)\, \tilde{h}(x)\, \tilde{w}(x)\, dx = 0, \qquad \text{all } u \in \tilde{U}.$$

Note that if $|Z(u^*)| = 0$, then we are finished since then $\tilde{U} = U_n$ and $u^*(x)\, \tilde{h}(x) \in U_n^* \backslash \{0\}$. Note also that the value of $\tilde{w}$ on $Z(u^*)$ has absolutely no effect on the above orthogonality condition since $\tilde{h}(x) = 0$ for all $x \in Z(u^*)$.

Let $U_n = \operatorname{span}\{u_1, \ldots, u_n\}$ where $u_1, \ldots, u_k$ is a basis for $\tilde{U}$. Set $\hat{U} = \operatorname{span}\{u_{k+1}, \ldots, u_n\}$. Thus $\dim \hat{U} = n - k$. Assume $n - k \geqslant 1$. The subspace $\hat{U}$ restricted to $\operatorname{int}(Z(u^*))$ is of dimension $n - k$. Otherwise there exists a $u \in \hat{U} \backslash \{0\}$ which vanishes identically on $\operatorname{int}(Z(u^*))$, and since $|Z(u^*)| = |\operatorname{int} Z(u^*)|$, we obtain $u \in \tilde{U}$, a contradiction. As such there exist points $x_1 < \cdots < x_{n-k}$, $x_j \in \operatorname{int} Z(u^*)$, $j = 1, \ldots, n - k$, such that

$$\det(u_i(x_j))_{i=k+1,\, j=1}^{n \quad n-k} \neq 0.$$

Since $x_j \in \operatorname{int}(Z(u^*))$, and $\operatorname{int}(Z(u^*))$ is open, there exist intervals (α_j, β_j), $j = 1, \ldots, n - k$, such that $x_j \in (\alpha_j, \beta_j) \subseteq [\alpha_j, \beta_j] \subseteq \operatorname{int} Z(u^*)$, the $[\alpha_j, \beta_j]$ are disjoint, and

$$\det\left(\int_{\alpha_j}^{\beta_j} u_i(x)\, dx\right)_{i=k+1,\, j=1}^{n \quad n-k} \neq 0.$$

Set $\int_0^1 u_i(x)\,\tilde{h}(x)\,dx = -\gamma_i$, $i=k+1,\dots,n$. There exist $\{c_j\}_{j=1}^{n-k}$ such that

$$\sum_{j=1}^{n-k} c_j \int_{\alpha_j}^{\beta_j} u_i(x)\,dx = \gamma_i, \qquad i=k+1,\dots,n.$$

We define $w \in W$ as follows.

(i) $w(x) = \tilde{w}(x)$, $x \in [0,1]\backslash \text{int}(Z(u^*))$,

(ii) $w(x) = |c_j|$, $x \in [\alpha_j, \beta_j]$, if $c_j \neq 0$,

(iii) $w(x) = \tilde{w}(x)$, $x \in [\alpha_j, \beta_j]$, if $c_j = 0$,

(iv) $w \in C[0,1]$ and $w(x) > 0$ for all $x \in [0,1]$.

This may be done. Now, set

$$h(x) = \begin{cases} \tilde{h}(x), & x \in [0,1]\backslash Z(u^*), \\ \operatorname{sgn} c_j, & x \in [\alpha_j, \beta_j], \\ 0, & \text{otherwise}. \end{cases}$$

By construction, it follows that

$$\int_0^1 u(x)\,h(x)\,w(x)\,dx = 0, \qquad \text{all } u \in U_n.$$

Our contradiction to the unicity property now follows. Set $f(x) = h(x)\,|u^*(x)|$. By construction, $f \in C(0,1)$, and we may alter the values at 0 and 1, if necessary, so that $f \in C[0,1]$. Furthermore, $|f(x)| = |u^*(x)|$ so that $f \in U_n^*\backslash\{0\}$. For any $u \in U_n$,

$$\begin{aligned} \|f-u\|_w &= \int_0^1 |f(x)-u(x)|\,w(x)\,dx \\ &\geqslant \int_0^1 (f(x)-u(x))\,h(x)\,w(x)\,dx \\ &= \int_0^1 f(x)\,h(x)\,w(x)\,dx \\ &= \|u^*\|_w \\ &= \|f\|_w. \end{aligned}$$

Thus the zero function is a best approximant to f from U_n in the L_w^1-norm. We now apply Theorem 2.3 to obtain our result. ∎

4. The A-Property

In this section we obtain specific conditions on U_n which are equivalent to the A-property. We defer, however, to the next section, the proof of the main result.

We first record the definitions and some properties of Tchebycheff (T-) and weak Tchebycheff (WT-) systems.

Definition 4.1. Let $U_n \subseteq C[0, 1]$ be an n-dimensional subspace. U_n is said to be a *Tchebycheff* (T-) *system* on $(0, 1)$ if no $u \in U_n \backslash \{0\}$ has more than $n-1$ zeros on $(0, 1)$.

Definition 4.2. U_n, as above, is said to be a *weak Tchebycheff* (WT-) *system* on $[0, 1]$ if no $u \in U_n$ has more than $n-1$ sign changes on $[0, 1]$. That is, there do not exist $n+1$ points $0 \leqslant x_1 < \cdots < x_{n+1} \leqslant 1$ and a $u \in U_n$ for which $u(x_i)\, u(x_{i+1}) < 0$, $i = 1,..., n$.

Both T- and WT- systems have various equivalent formulations. Two of these for WT-systems are contained in the following proposition.

Proposition 4.1 (Jones and Karlovitz [7]). *U_n, as above, is a* WT-*system on* $[0, 1]$ *if and only if any one of the following hold.*

(1) *Given* $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ *there exists a* $u \in U_n \backslash \{0\}$ *for which*

$$(-1)^i u(x) \geqslant 0, \qquad x \in [x_{i-1}, x_i], \quad i = 1,..., n.$$

(2) *If* $U_n = \operatorname{span}\{u_1,..., u_n\}$ *then there exists an* $\varepsilon \in \{-1, 1\}$ *such that for any choice of* $0 \leqslant x_1 < \cdots < x_n \leqslant 1$

$$\varepsilon \det(u_i(x_j))_{i,j=1}^n \geqslant 0.$$

Various additional properties of T- and WT-systems will be employed. Before stating these properties we present the following simple lemma which is used in the proofs of Proposition 4.3 and Lemma 4.8. This lemma is stated in Stockenberg [22], but the proof therein is in error.

Lemma 4.2. *Let U be a subspace of* $C[0, 1]$. *Let* $x_1,..., x_m \in [0, 1]$, *m finite, be points such that for each* $i = 1,..., m$, *there exists a* $u_i \in U$ *such that* $u_i(x_i) \neq 0$. *Then there exists a* $u \in U$ *for which* $u(x_i) \neq 0$, $i = 1,..., m$.

Proof. We prove the lemma via induction on the number of points. There exists a $u_1 \in U$ for which $u_1(x_1) \neq 0$. Assume that given $x_1,..., x_{k-1}$ there exists a $u^* \in U$ for which $u^*(x_i) \neq 0$, $i = 1,..., k-1$. If $u^*(x_k) \neq 0$ there is nothing to prove. Assume $u^*(x_k)=0$. There exists a $u_k \in U$ for which $u_k(x_k) \neq 0$. Let $u = u^* + \varepsilon u_k$ for ε sufficiently small, $\varepsilon \neq 0$. Then necessarily $u(x_i) \neq 0$, $i = 1, \ldots, k$. ▮

The following three known results will be used.

PROPOSITION 4.3 (Stockenberg [22]). *Let U_n be a* WT-*system on* $[0, 1]$. *Assume that for each $x \in (0, 1)$ there exists a $u \in U_n$ for which $u(x) \neq 0$. Assume also that no $u \in U_n \backslash \{0\}$ vanishes on a subinterval of* $[0, 1]$. *Then U_n is a* T-*system on* $(0, 1)$.

PROPOSITION 4.4 (Sommer [16]). *Let U_n be a* WT-*system of dimension n on* $[0, 1]$. *For any $0 \leqslant a < b \leqslant 1$, $U_n|_{[a,b]}$ is a* WT-*system of dimension $\leqslant n$.*

PROPOSITION 4.5 (Sommer and Strauss [19], Stockenberg [22]). *Let U_n be a* WT-*system of dimension n on* $[0, 1]$. *Then there exists a $U_{n-1} \subseteq U_n$ such that U_{n-1} is a* WT-*system of dimension $n-1$ on* $[0, 1]$.

The following proposition will not be used in this work. However, it is sufficiently simple and elegant to present here.

PROPOSITION 4.6 (Sommer [18, Theorem 6]). *Let $U_n \subseteq C[0, 1]$ be an n-dimensional subspace which satisfies the* A-*property. Then U_n is a* WT-*system on* $[0, 1]$.

Proof. Let $0 = x_0 < x_1 < \cdots < x_n = 1$. By Proposition 4.1, it suffices to prove the existence of a $u \in U_n \backslash \{0\}$ such that $(-1)^i u(x) \geqslant 0$, $x \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$. Since dim $U_n = n$, there exists a $\tilde{u} \in U_n \backslash \{0\}$ such that $\tilde{u}(x_i) = 0$, $i = 1, \ldots, n-1$, i.e., $x_i \in Z(\tilde{u})$, $i = 1, \ldots, n-1$. From the A-property it therefore follows that there exists a $u \in U_n \backslash \{0\}$ such that $(-1)^i u(x) \geqslant 0$, $x \in [x_{i-1}, x_1]$, $i = 1, \ldots, n$. That is, if I_j, $j = 1, \ldots, m$, are the maximal open intervals of $(0, 1)$ on which $\tilde{u}(x)$ does not vanish, then $x_i \notin I_j$, $i = 1, \ldots, n-1$; $j = 1, \ldots, m$, so that we may choose the $\varepsilon_j \in \{-1, 1\}$, as in Definition 2.1′, to ensure our result. ▮

Our analysis of the A-property is based on the following theorem, the proof of which is deferred to the next section. In this section we will consider various consequences of this theorem.

THEOREM 4.7. *Let U_n be an n-dimensional subspace of* $C[0, 1]$. *Assume that U_n satisfies the* A-*property. Given $\tilde{u} \in U_n \backslash \{0\}$, let*

$$\tilde{U} = \{u: u \in U_n, u(x) = 0 \text{ a.e. on } Z(\tilde{u})\}.$$

Then the number of maximal open intervals of $(0, 1)$ *on which* $\tilde{u}(x)$ *does not vanish is bounded above by* $\dim \tilde{U}$.

This immediately implies that U_n is a WT-system, as is any $\tilde{U}$ as above. Throughout the rest of this section we assume that Theorem 4.7 holds.

LEMMA 4.8. *Let* $A = \{x: x \in [0, 1], u(x) = 0$ *for all* $u \in U_n\}$, *and let* $B = (0, 1) \backslash A$. *Then* B *is the union of at most* n *open intervals.*

Remark. We call A the *fundamental zero set* of U_n.

Proof. Assume not. Then we can find points $y_1 < \cdots < y_{n+1}$, $y_i \in B$, and $x_i \in (y_i, y_{i+1})$, $i = 1,..., n$, for which $x_i \in A$. From Lemma 4.2, there exists a $u \in U_n$ such that $u(y_i) \neq 0$, $i = 1,..., n+1$. Since u must vanish at the x_i's which interlace the y_i's, this implies the existence of at least $n+1$ maximal open intervals of (0, 1) on which $u(x)$ does not vanish. This contradicts Theorem 4.7. ∎

Therefore $B = \bigcup_{i=1}^{k} (a_i, b_i)$ where $k \leqslant n$, and the (a_i, b_i) are disjoint.

PROPOSITION 4.9. *Let* $U_n^i = U_n|_{[a_i, b_i]}$, $i = 1,..., k$. *Then* $U_n = U_n^1 \oplus \cdots \oplus U_n^k$. *That is, each* U_n^i *has a basis of functions which vanish identically off* $[a_i, b_i]$, *and* $n = \sum_{i=1}^{k} \dim U_n^i$.

Proof. If $k = 1$, there is nothing to prove. Assume $k > 1$. For some i, set $(a, b) = (a_i, b_i)$, and $V = U_n^i$. Then $\dim V = r \geqslant 1$. We claim that $r < n$. If $\dim V = n$, then there exist points $a < x_1 < \cdots < x_n < b$ and $u^* \in V$ for which $u^*(x_i)(-1)^i > 0$, $i = 1,..., n$. Since $u^*(a) = 0$ if $a > 0$ and $u^*(b) = 0$ if $b > 1$ (and at least one of the conditions hold since $k > 1$) it follows that there exist at least n (and thus exactly n by Theorem 4.7) maximal open intervals on which u^* does not vanish, all of which are in (a, b). As such it is necessary that $u^*(x) = 0$ for all $x \notin [a, b]$. Now, set

$$U(u^*) = \{u: u(x) = 0 \text{ a.e. on } Z(u^*)\}.$$

Since $k > 1$, $\dim U(u^*) < n$. This contradicts Theorem 4.7 since u^* has n maximal open intervals on which u^* does not vanish. Thus $r < n$.

It remains to prove that there is a basis for V all of whose elements vanish identically off $[a, b]$. Since $1 \leqslant r < n$, there exist $n - r$ linearly independent functions in U_n which vanish identically on $[a, b]$. Set

$$W = \{u: u \in U_n, u(x) \equiv 0, x \in [a, b]\}.$$

W is a subspace of U_n, and $\dim W = n - r$. Let $V = \text{span}\{u_1,..., u_r\}$, where we choose each u_i with at least $r - 1$ sign changes on (a, b), i.e., each u_i has associated $I_1,..., I_m$ in (a, b) with $m \geqslant r$.

We claim that u_i on $[0, 1]\backslash[a, b]$ is an element of W. If not then $W \cup \{u_i\}$ on $[0, 1]\backslash[a, b]$ is a subspace of dimension $n-r+1$. As such there exists a $w \in W$ and a constant α such that $\alpha u_i + w$ changes sign $n-r$ times on $[0, 1]\backslash[a, b]$, i.e., associated with $\alpha u_i + w$ are $I'_1, ..., I'_k$ with $k \geqslant n-r+1$. If $\alpha \neq 0$, we immediately obtain a contradiction to Theorem 4.7 since $m+k \geqslant n+1$, and $(\alpha u_i + w)(a) = 0$ if $a > 0$, $(\alpha u_i + w)(b) = 0$ if $b < 1$. Moreover we may assume $\alpha \neq 0$ since a small perturbation of α will not decrease the number of sign changes in $[0, 1]\backslash[a, b]$.

Since u_i on $[0, 1]\backslash[a, b]$ is an element of W for each $i = 1, ..., r$, it follows that there exist $w_i \in W$, $i = 1, ..., r$, such that $(u_i - w_i)(x) = 0$ for all $x \notin [a, b]$, and $(u_i - w_i)(x) = u_i(x)$ for all $x \in [a, b]$. This proves the proposition. ∎

On the basis of the above proposition, we can and will assume that for each $x \in (0, 1)$ there exists a $u \in U_n$ for which $u(x) \neq 0$. Under these assumptions we first deal with the simplest case.

PROPOSITION 4.10. *If no* $u \in U_n\backslash\{0\}$ *vanishes on a subinterval of* $(0, 1)$ *then* U_n *is a* T-*system on* $(0, 1)$.

Remark. This is the result due to Havinson [5], see also Kroó [10].

Proof. A consequence of Proposition 4.3 and Theorem 4.7. ∎

To deal with the remaining case we first present the following definition.

DEFINITION 4.3. We say that $[a, b]$, $0 \leqslant a < b \leqslant 1$, is a *zero interval* of $u \in U_n$ if $u(x) \equiv 0$ for $x \in [a, b]$, and $u(x) \neq 0$ for all $x \in (a-\varepsilon, a)$, some $\varepsilon > 0$, if $a > 0$, and $u(x) \neq 0$ for all $x \in (b, b+\varepsilon)$, some $\varepsilon > 0$ if $b < 1$.

Note that the A-property implies that the zero set of each $u \in U_n$ is composed of at most $n+1$ distinct points and/or intervals.

LEMMA 4.11. *There exist at most* $n-1$ *points* $0 < b_1 < \cdots < b_r < 1$ $(r \leqslant n-1)$ *such that* $[0, b_i]$ *is a zero interval of some* $u \in U_n\backslash\{0\}$.

Proof. Assume to the contrary that there exist points $0 < b_1 < \cdots < b_n < 1$ such that $[0, b_i]$ is a zero interval of $u_i \in U_n\backslash\{0\}$, $i = 1, ..., n$.

Set $b_0 = 0$, $b_{n+1} = 1$, and choose $x_i \in (b_i, b_{i+1})$ such that $u_i(x_i) \neq 0$, $i = 0, 1, ..., n$. (Note that for any $x_0 \in (0, b_1)$ there exists a $u_0 \in U_n$ for which $u_0(x_0) \neq 0$.) By the above, such x_i exist. Then $(u_i(x_j))_{i,j=0}^n$ is an $(n+1) \times (n+1)$ triangular non-singular matrix since its diagonal entries are non-zero. But U_n is n-dimensional, a contradiction. ∎

In a totally similar fashion we have

LEMMA 4.12. *There exist at most* $n-1$ *points* $0<a_1<\cdots<a_s<1$ $(s\leqslant n-1)$ *such that* $[a_i, 1]$ *is a zero interval of some* $u\in U_n\backslash\{0\}$.

PROPOSITION 4.13. *Let* $u^*\in U_n\backslash\{0\}$ *have a zero interval* $[a,b]$, $0<a<b<1$. *Then* $a=a_i$ *and* $b=b_j$ *for some* $i=1,\ldots,s$; $j=1,\ldots,r$. *Furthermore there exists a* $v\in U_n$ *such that*

$$v(x)=\begin{cases} u^*(x), & 0\leqslant x\leqslant a,\\ 0, & a\leqslant x\leqslant 1,\end{cases}$$

and a $w\in U_n$ *such that*

$$w(x)=\begin{cases} 0, & 0\leqslant x\leqslant b,\\ u^*(x), & b\leqslant x\leqslant 1.\end{cases}$$

Proof. Let u^* be as above and set

$$U(u^*)=\{u: u\in U_n, u(x)=0 \text{ a.e. on } Z(u^*)\}.$$

$U(u^*)$ is a subspace of U_n of dimension k, $1\leqslant k<n$, which satisfies the A-property. Furthermore the fundamental zero set of $U(u^*)$ contains the interval $[a,b]$. As such we can apply Proposition 4.9 to $U(u^*)$ to obtain $v, w\in U(u^*)\subseteq U_n$ satisfying the above conditions. Since $v(x)=0$ for all $x\in[a,1]$ and $v(x)=u^*(x)\neq 0$ for $x\in(a-\varepsilon, a)$, some $\varepsilon>0$, it necessarily follows that $a=a_i$ for some $i=1,\ldots,s$. Similarly $b=b_j$ for some $j=1,\ldots,r$. ∎

Let $\{c_1,\ldots,c_k\}$ denote the ordered distinct points of the set $\{b_1,\ldots,b_r, a_1,\ldots,a_s\}$ and set $c_0=0$, $c_{k+1}=1$. By the previous proposition, if $u\in U_n$ has a zero interval $[a,b]$, then $a=c_i$, $b=c_j$ for some $0\leqslant i<j\leqslant k+1$.

PROPOSITION 4.14. $U_n|_{(c_{i-1},c_i)}$ *is a* T-*system for* $i=1,\ldots,k+1$.

Proof. From Theorem 4.7, U_n is a WT-system on $[0,1]$. $U_n|_{[c_{i-1},c_i]}$ is a WT-system by Proposition 4.4. By Proposition 4.3, it follows that $U_n|_{(c_{i-1},c_i)}$ is a T-system. ∎

We deduce one additional property of U_n. For $0\leqslant i<j\leqslant k+1$, set

$$V_{ij}=\{u: u\in U_n, u(x)=0, x\in[0,c_i)\cup(c_j,1]\}.$$

PROPOSITION 4.15. *If* $\dim V_{ij}>0$, *then* V_{ij} *is a* WT-*system.*

Proof. Let $\dim V_{ij}=m_{ij}>0$. Assume that V_{ij} is not a WT-system. Then by definition there exists a $u^*\in V_{ij}$ with at least m_{ij} sign changes on (c_i,c_j). Thus there exist at least $m_{ij}+1$ maximal open intervals in (c_i,c_j) on which $u^*(x)$ does not vanish. Set

$$U(u^*)=\{u: u\in U_n, u(x)=0 \text{ a.e. on } Z(u^*)\}.$$

Since $[0, c_i) \cup (c_j, 1] \subseteq Z(u^*)$, it follows that $U(u^*) \subseteq V_{ij}$. Thus by Theorem 4.7

$$m_{ij} + 1 \leqslant \dim U(u^*) \leqslant \dim V_{ij} = m_{ij},$$

a contradiction. ▮

For ease of exposition let us rerecord the various properties of U_n which result from Theorem 4.7 and the A-property.

(a) Let $B = \{x\colon x \in (0, 1), \exists u \in U_n \text{ such that } u(x) \neq 0\}$. Then $B = \bigcup_{i=1}^{k} (a_i, b_i)$ where $k \leqslant n$ and the (a_i, b_i) are disjoint. Furthermore, $U_n = U_n^1 \oplus \cdots \oplus U_n^k$, where $U_n^i = U_n|_{[a_i, b_i]}$, $i = 1,..., k$.

This immediately implies that the approximation problem on $[0, 1]$ is really k distinct, independent approximation problems on $[a_i, b_i]$, $i = 1,..., k$. As such we might as well assume that $B = (0, 1)$.

In this case,

(b)(1) U_n is a WT-system on $(0, 1)$.

(2) There exist points $c_0 = 0 < c_1 < \cdots < c_k < c_{k+1} = 1$ $(k \leqslant 2n - 2)$ such that $U_n|_{(c_{i-1}, c_i)}$ is a T-system, $i = 1,..., k + 1$.

(3) If $[a, b]$ is a zero interval of $u \in U_n \backslash \{0\}$, then $a = c_i$, $b = c_j$ for some $0 \leqslant i < j \leqslant k + 1$, and

(i) there exists a $v \in U_n$ for which

$$v(x) = \begin{cases} u(x), & 0 \leqslant x < a, \\ 0, & a \leqslant x \leqslant 1, \end{cases}$$

(ii) there exists a $w \in U_n$ for which

$$w(x) = \begin{cases} 0, & 0 \leqslant x \leqslant b, \\ u(x), & b < x \leqslant 1. \end{cases}$$

(4) If $V_{ij} = \{u\colon u \in U_n, u(x) = 0, x \in [0, c_i) \cup (c_j, 1]\}$ for $0 \leqslant i < j \leqslant k + 1$, then V_{ij} is a WT-system of dim V_{ij}.

Note that (1) is actually contained in (4).

To complete the picture we prove that these conditions imply the A-property. Without loss of generality, we will assume that we are in case (b).

THEOREM 4.16. *Let U_n be an n-dimensional subspace of $C[0, 1]$. Assume that for each $x \in (0, 1)$ there exists a $u \in U_n$ for which $u(x) \neq 0$. If U_n satisfies conditions* (1)–(4) *as above, then U_n satisfies the* A-*property.*

Proof. Let $u^* \in U_n \backslash \{0\}$. We divide the proof into three cases.

Case 1. u^* has no zero intervals. Let $0 < x_1 < \cdots < x_r < 1$ denote the zeros of u^* in $(0, 1)$. Since U_n is a WT-system, then it follows by a result of Stockenberg [22, Theorem 1] that $r \leqslant n-1$. Choose $\varepsilon_i \in \{-1, 1\}$, $i = 1,..., r+1$. We must exhibit a $u \in U_n \backslash \{0\}$ for which $\varepsilon_i u(x) \geqslant 0$, $x \in (x_{i-1}, x_i)$, $i = 1,..., r+1$, where $x_0 = 0$, $x_{r+1} = 1$.

From the sequence $(\varepsilon_1,..., \varepsilon_{r+1})$ form the sequence $(\delta_1,..., \delta_k)$ where $\delta_i = \varepsilon_1(-1)^{i+1}$, $i = 1,..., k$, and the number of sign changes in the two sequences $(\varepsilon_1,..., \varepsilon_{r+1})$ and $(\delta_1,..., \delta_k)$ are the same. Thus $k \leqslant r+1 \leqslant n$. Set $I_1 = (x_0, x_{i_1})$ where $\varepsilon_1 = \cdots = \varepsilon_{i_1}$, $\varepsilon_{i_1}\varepsilon_{i_1+1} = -1$. Set $I_2 = (x_{i_1}, x_{i_2})$ where $\varepsilon_{i_1+1} = \cdots = \varepsilon_{i_2}$, $\varepsilon_{i_2}\varepsilon_{i_2+1} = -1$, etc., to obtain $I_1, I_2,..., I_k$. From Proposition 4.5, there exists a subspace U_k of dim k of U_n such that U_k is a WT-system on $[0, 1]$. From Proposition 4.1 there exists a $u \in U_k \subseteq U_n$, $u \not\equiv 0$, such that $\delta_i u(x) \geqslant 0$, $x \in I_i$, $i = 1,..., k$. This is our required function.

We will therefore assume that u^* has zero intervals. Each zero interval is, by (3), of the form $[c_{i_m}, c_{j_m}]$, $m = 1,..., p$, where $0 \leqslant i_1 < j_1 < i_2 < \cdots < j_p \leqslant k+1$. Set $J = \bigcup_{m=1}^{p} (c_{i_m}, c_{j_m})$.

Case 2. $(0, c_1) \nsubseteq J$ or $(c_k, 1) \nsubseteq J$. Assume, without loss of generality, that $(0, c_1) \nsubseteq J$. Then, by (3), there exists a $v \in U_n$ for which

$$v(x) = \begin{cases} u^*(x), & 0 \leqslant x \leqslant c_{i_1}, \\ 0, & c_{i_1} \leqslant x \leqslant 1. \end{cases}$$

Since u^* has no zero interval in $[0, c_{i_1}]$, v has no zero interval in $[0, c_{i_1}]$. Furthermore $v \in V_{0,i_1}$, i.e., $v(x) \equiv 0$, $x \in (c_{i_1}, 1]$. By (4), v is an element of the WT-system V_{0,i_1}. We now apply the reasoning of Case 1 to the interval $(0, c_{i_1})$ and the subspace V_{0,i_1}.

Case 3. $(0, c_1) \subseteq J$ and $(c_k, 1) \subseteq J$. Then u^* has no zero interval in $[c_{j_1}, c_{i_2}]$ where $0 < j_1 < i_2 < k+1$. By (3) there exists a $v \in U_n$ for which

$$v(x) = \begin{cases} u^*(x), & 0 \leqslant x \leqslant c_{i_2}, \\ 0, & c_{i_2} \leqslant x \leqslant 1. \end{cases}$$

Since $v(x) = 0$ for $x \in (0, c_{j_1})$, it follows that $v \in V_{j_1,i_2}$, and has no zero interval in $[c_{j_1}, c_{i_2}]$. Again we apply the reasoning of Case 1 since, by (4), V_{j_1,i_2} is a WT-system. ∎

The above theorem should be contrasted with work of Sommer [16, 17]. He presents slightly different conditions which imply the A-property.

5. Proof of Theorem 4.7

For convenience we restate the result to be proved in this section.

THEOREM 4.7. *Let U_n be an n-dimensional subspace of* $C[0, 1]$. *Assume that U_n satisfies the* A-*property. Given $\tilde{u} \in U_n \backslash \{0\}$, set*

$$\tilde{U} = \{u: u \in U_n, u(x) = 0 \text{ a.e. on } Z(\tilde{u})\}.$$

Then the number of maximal open intervals of $(0, 1)$ *on which $\tilde{u}(x)$ does not vanish is bounded above by* $\dim \tilde{U}$.

The proof of this theorem is lengthy. As such it is divided into a series of steps. The proof is by induction and we therefore first prove the case $n = 1$.

LEMMA 5.1. *Theorem* 4.7 *holds for* $n = 1$.

Proof. Let $U_1 = \text{span}\{u\}$. Assume that there exists $I_1 = (a, b)$ and $I_2 = (c, d)$, $a < b \leqslant c < d$, such that $u(x)$ has strict sign ε_1 on (a, b), ε_2 on (c, d), and vanishes identically on $[b, c]$. By the A-property, there exists a $\tilde{u} \in U_1 \backslash \{0\}$ such that $\varepsilon_1 \tilde{u}(x) \geqslant 0$, $x \in (a, b)$, and $(-\varepsilon_2)\, \tilde{u}(x) \geqslant 0$, $x \in (c, d)$. Since $\tilde{u} = \alpha u$, this is impossible. ∎

Associated with each $u \in U_n \backslash \{0\}$, let $I_1, ..., I_m$ (m may be infinite) denote all the maximal open intervals of $(0, 1)$ on which u does not vanish. For given $u \in U_n \backslash \{0\}$, let $m(u)$ denote the number of such intervals.

PROPOSITION 5.2. *If for every $u \in U_n \backslash \{0\}$, $m(u) \leqslant M$ (M some finite constant), then Theorem* 4.7 *holds.*

Proof. Let $u^* \in U_n \backslash \{0\}$ be such that $\max\{m(u): u \in U_n\} = m(u^*) = M$. Set

$$U^* = \{u: u \in U_n, u(x) = 0 \text{ a.e. on } Z(u^*)\}.$$

If $\dim U^* < n$, then by induction we may assume that $m(u^*) = M \leqslant \dim U^*$. We therefore assume that $U^* = U_n$, and $M \geqslant n + 1$. Let $I_1^*, ..., I_M^*$ denote the maximal open intervals of $(0, 1)$ on which u^* does not vanish. Since M is finite, we may, for convenience, assume that $I_1^*, ..., I_M^*$ are in increasing order, i.e., for $x \in I_i^*$, $y \in I_j^*$, $x < y$ if $i < j$.

Set

$$\Sigma_M = \{\boldsymbol{\varepsilon} = (\varepsilon_1, ..., \varepsilon_M): \varepsilon_i \in \{-1, 1\}, i = 1, ..., M\}$$

and let $\boldsymbol{\varepsilon}^* \in \Sigma_M$ be such that

$$\varepsilon_i^* u^*(x) > 0, \qquad x \in I_i^*, \quad i = 1, ..., M.$$

Set $U_n = \text{span}\{u_1,..., u_n\}$ and define

$$d_{ij} = \int_{I_j^*} u_i(x)\,dx, \qquad i=1,..., n; \quad j=1,..., M.$$

Since $M > n$, there exists a vector $\mathbf{c} = (c_1,..., c_M) \neq \mathbf{0}$ for which

$$\sum_{j=1}^{M} d_{ij} c_j = 0, \qquad i=1,..., n.$$

Thus

$$\sum_{j=1}^{M} c_j \int_{I_j^*} u(x)\,dx = 0, \qquad \text{all } u \in U_n. \tag{5.1}$$

Choose $\boldsymbol{\varepsilon} \in \Sigma_M$ such that $\varepsilon_j = \text{sgn}\, c_j$ if $c_j \neq 0$, and let ε_j alternate in sign on the indices j for which $c_j = 0$, i.e., if $c_{j_1} = c_{j_2} = 0$, and $c_j \neq 0$ for all $j_1 < j < j_2$, then $\varepsilon_{j_1}\varepsilon_{j_2} = -1$.

By the A-property there exists a $u_\varepsilon \in U_n \backslash \{0\}$ for which

(i) $\varepsilon_j u_\varepsilon(x) \geqslant 0$, $x \in I_j^*$, $j=1,..., M$,

(ii) $u_\varepsilon(x) = 0$ a.e. on $Z(u^*)$.

(Condition (ii) has no significance here since $U^* = U_n$.) From (i) and the choice of $\boldsymbol{\varepsilon}$, $(\text{sgn}\, c_j) \int_{I_j^*} u_\varepsilon(x)\,dx \geqslant 0$ for all j. Thus from (5.1) it follows that $u_\varepsilon(x) = 0$ for all $x \in I_j^*$ if $c_j \neq 0$. If all the c_j are non-zero or if no $u \in U_n \backslash \{0\}$ vanishes on a set of positive measure then we immediately arrive at a contradiction. We therefore assume that some (but not all since $\mathbf{c} \neq \mathbf{0}$) of the c_j's are zero.

Let $I_{r_1}^*,..., I_{r_k}^*$, $k < M$, be such that there exists an $x_{r_i} \in I_{r_i}^*$, $i=1,..., k$, for which $u_\varepsilon(x_{r_i}) \neq 0$. Let $I_{r'_1}^*,..., I_{r'_{M-k}}^*$ denote the complementary set to $\{I_{r_i}^*\}_{i=1}^k$ in $\{I_i^*\}_{i=1}^M$. By the choice of $\boldsymbol{\varepsilon}$, $u_\varepsilon(x)$ has a zero between $I_{r_i}^*$ and $I_{r_{i+1}}^*$, $i=1,..., k-1$. Set

$$U_\varepsilon = \{u\colon u \in U_n,\ u(x) = 0 \text{ a.e. on } Z(u_\varepsilon)\}.$$

Since $Z(u_\varepsilon)$ contains $\{I_{r'_i}^*\}_{i=1}^{M-k}$, $u^* \notin U_\varepsilon$, and $\dim U_\varepsilon < n$. Thus by the induction hypothesis, $k \leqslant \dim U_\varepsilon = s$.

Case 1. $u^* = u$ on $\bigcup_{i=1}^k I_{r_i}^*$ for some $u \in U_\varepsilon$.

In this case $(u^* - u)(x) = 0$ for all $x \in \bigcup_{i=1}^k I_{r_i}^*$ while $(u^* - u)(x) = u^*(x)$ for all $x \notin \bigcup_{i=1}^k I_{r_i}^*$. Set

$$\tilde{U} = \{u\colon u \in U_n,\ u(x) = 0 \text{ a.e. on } Z(u^* - u)\}.$$

$u^* \notin \tilde{U}$ and thus by the induction hypothesis on $\tilde{U}$, and since $(u^* - u)(x) = u^*(x)$ on $\bigcup_{i=1}^{M-k} I_{r_i'}^*$, it follows that $M - k \leqslant \dim \tilde{U}$. Thus $M = k + M - k \leqslant \dim U_\varepsilon + \dim \tilde{U} \leqslant \dim U_n = n$ since $U_\varepsilon \cap \tilde{U} = \{0\}$. Thus $M \leqslant n$, a contradiction.

Case 2. $u^* \neq u$ on $\bigcup_{i=1}^{k} I_{r_i}^*$ for any $u \in U_\varepsilon$.

Let $V = U_\varepsilon \cup \{u^*\}$. Since $u^* \notin U_\varepsilon$ restricted to $\bigcup_{i=1}^{k} I_{r_i}^*$, and $\dim U_\varepsilon = s \geqslant k$, it follows that there exist $s+1$ ordered points $\{x_i\}_{i=1}^{s+1}$ in $\bigcup_{i=1}^{k} I_{r_i}^*$ and a $v \in V$ for which $v(x_i)(-1)^i > 0$, $i = 1,..., s+1$. Let $v(x) = \alpha u^*(x) + u(x)$ where $u \in U_\varepsilon$. If $\alpha = 0$, then it is easily seen that we contradict the induction hypothesis. The function $v \in V \subseteq U_n$ has at least $s+1$ maximal open intervals in $\overline{\bigcup_{i=1}^{k} I_{r_i}^*}$ on which v does not vanish. These intervals all lie in $\bigcup_{i=1}^{k} I_{r_i}^*$ since $v(x) = \alpha u^*(x)$ on $\bigcup_{i=1}^{M-k} I_{r_i'}^*$ and $\alpha u^*(x)$ vanishes at the endpoints of each $I_{r_i}^*$ in $(0, 1)$. Thus $v(x)$ on $(0, 1)$ has a total of at least $M - k + s + 1 \geqslant M + 1$ (recall that $s \geqslant k$) maximal open intervals on which $v(x)$ does not vanish. This contradicts the maximality hypothesis on M. ∎

It remains to consider the case where there is no uniform bound on the number of maximal open intervals of $(0, 1)$ on which $u(x)$ does not vanish for all $u \in U_n$. This case is technically the more difficult since we cannot apply the reasoning of Case 2 of the above proposition.

PROPOSITION 5.3. *Theorem* 4.7 *is valid.*

Proof. We assume that there is no uniform bound on the number of maximal open intervals of $(0, 1)$ on which $u(x)$ does not vanish for all $u \in U_n$. The proof in this case is also by induction. Lemma 5.1 covers the case $n = 1$. We therefore assume that $n > 1$ and that we are given $u^* \in U_n \backslash \{0\}$ with m (m may be infinite) maximal open intervals $I_1^*,..., I_m^*$ of $(0, 1)$ on which $u^*(x)$ does not vanish. By our hypothesis we may assume that m is as large as is necessary. We also assume, from the induction hypothesis, that $U^* = U_n$, where

$$U^* = \{u: u \in U_n, u(x) = 0 \text{ a.e. on } Z(u^*)\}$$

As in the proof of Proposition 5.2, let $\varepsilon_i^* \in \{-1, 1\}$, $i = 1,..., m$, be such that

$$\varepsilon_i^* u^*(x) > 0, \qquad x \in I_i^*, \quad i = 1,..., m.$$

Given any set of $n+1$ I_i^*'s, say, $I_1^*,..., I_{n+1}^*$, there exists a $\tilde{\mathbf{c}} = (\tilde{c}_1,..., \tilde{c}_{n+1}) \neq \mathbf{0}$ for which

$$\sum_{j=1}^{n+1} \tilde{c}_j \int_{I_j^*} u(x)\, dx = 0, \qquad \text{all } u \in U_n.$$

Let $\mathbf{c} = (c_1, ..., c_m)$ where $c_i = \tilde{c}_i$, $i = 1, ..., n+1$, and $c_i = 0$, $i \geqslant n+2$. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, ..., \varepsilon_m)$, $\varepsilon_i \in \{-1, 1\}$, where

(1) $\varepsilon_j = \operatorname{sgn} c_j$ if $c_j \neq 0$,

(2) ε_j alternate in sign on adjacent I_j^* for which $c_j = 0$.

Since U_n satisfies the A-property, there exists a $u^1 \in U_n \backslash \{0\}$ for which $\varepsilon_j u^1(x) \geqslant 0$, $x \in I_j^*$, $j = 1, ..., m$, and $u^1(x) = 0$ a.e. on $Z(u^*)$. Since

$$\sum_{j=1}^{m} c_j \int_{I_j^*} u^1(x)\, dx = 0,$$

it follows that $u^1(x) = 0$ on I_j^* if $c_j \neq 0$. Furthermore, from the choice of $\boldsymbol{\varepsilon}$, we may assume that between each I_i^* and I_j^*, $i \neq j$, $u^1(x)$ vanishes.

Let $i_1, ..., i_k \in \{1, ..., m\}$ be such that $u^1(x)$ does not vanish identically on $I_{i_j}^*$, $j = 1, ..., k$, and does vanish identically off $\bigcup_{j=1}^k I_{i_j}^*$. Set

$$U(u^1) = \{u \colon u \in U_n,\ u(x) = 0 \text{ a.e. on } Z(u^1)\}.$$

(This notation will be used throughout.)

Since $\mathbf{c} \neq \mathbf{0}$, we have $u^* \notin U(u^1)$, and $k_1 = \dim U(u^1) \leqslant n-1$. By the induction hypothesis $k \leqslant k_1$, and we can also apply to $U(u^1)$ all the results of Section 4.

In particular, let M denote the interior of the support of $U(u^1)$ in $(0, 1)$, i.e., for each $x \in M$ there exists a $u \in U(u^1)$ for which $u(x) \neq 0$. By Lemma 4.8 and Proposition 4.14, it follows that there exist intervals $(a_i, b_i)_{i=1}^r$ $(r \leqslant 2n)$ such that $\bigcup_{i=1}^r (a_i, b_i) \subseteq M \subseteq \overline{\bigcup_{i=1}^r (a_i, b_i)}$, and $U(u^1)|_{(a_i, b_i)}$ is a T-system of dimension $m_i \geqslant 1$, for each $i = 1, ..., r$.

Case I. There exists a subinterval (a, b) of M such that $u^*|_{(a,b)} \in U(u^1)|_{(a,b)}$.

Let $u \in U(u^1)$ be such that $(u^* - u)(x) = 0$, $x \in (a, b)$. Let $U(u^* - u)$ be defined analogously to $U(u^*)$. Since $u^* \notin U(u^* - u)$, it follows that $\dim U(u^* - u) \leqslant n-1$. Off M, $(u^* - u)(x) = u^*(x)$. Thus $(u^* - u)(x)$ has at least $m - k$ maximal open intervals of $(0, 1)$ on which $(u^* - u)(x)$ does not vanish. Thus by the induction hypothesis $m - k \leqslant \dim U(u^* - u) \leqslant n - 1$. Since $k \leqslant n-1$, it follows that $m \leqslant 2(n-1)$. This contradicts our assumption on m.

Case II. $u^*|_{(a,b)} \notin U(u^1)|_{(a,b)}$ on any subinterval of M.

We start with (a_1, b_1). $U(u^1)|_{(a_1, b_1)}$ is a T-system of dimension m_1. For notational ease, set $u_1^*(x) = u^*(x)$. Since $u_1^*|_{(a_1, b_1)} \notin U(u^1)|_{(a_1, b_1)}$ there exists $u_2^*(x) = \alpha u_1^*(x) - u(x)$, $u \in U(u^1)$, such that $u_2^*(x)$ has at least m_1 sign changes on (a_1, b_1). If $\alpha = 0$, then we contradict the fact that $u \in U(u^1)$. Thus we may assume $\alpha = 1$. Since $u \in U(u^1)$, $u_2^*(x) = u_1^*(x)$ off M. There exists $m-k$ ($>n$) maximal open intervals on $(0, 1)$ (off M) on which u_2^* does not vanish. As such we can construct a function u^2 with respect to u_2^* as u^1 was constructed with respect to u_1^*.

The object of this exercise is to show that if we repeat this process a sufficient, but finite, number of times, with care, we will eventually arrive at a situation where $U(u^2) \not\subseteq U(u^1)$. This is desired for the following reason.

Case II(i). $U(u^2) \not\subseteq U(u^1)$.

This implies that the support of u^2 is not contained in the support of u^1. Thus there exists a $u_1 \in U(u^1)$ and a $u_2 \in U(u^1)$ for which $(u_1+u_2)(x) \neq 0$ a.e. off $Z(u^1) \cap Z(u^2)$. It is easily seen that $u_2^* \notin U(u_1+u_2)$. Thus by the induction hypothesis, and since $U(u_1+u_2)$ contains both $U(u^1)$ and $U(u^2)$, it follows that $\dim U(u^1) < \dim U(u_1+u_2) \leqslant n-1$. We now replace u^1 by u_1+u_2, and $u_1^*(x)$ by $u_2^*(x)$, and start the process again. The exact construction of $u_2^*(x)$ and $u^2(x)$, as above, is unimportant. What is important is that $u_2^*(x) = u^*(x)$ on $Z(u^1) \cap Z(u^2)$ and $u^*(x)$, thereon, has many maximal open intervals on which it does not vanish. If we can repeat the above process a sufficient number of times, then we must eventually arrive at a contradiction.

Case II(ii). $U(u^2) \subseteq U(u^1)$.

We now construct an algorithm which shows that we must eventually arrive at Case II(i). In this algorithm we construct a sequence of μ_j^*, $j=1, 2, \ldots$. It is important to note that if we do not revert to Case II(i), then $u_j^*(x) = u^*(x)$ for all j, off M. We let $\boldsymbol{\mu}^j = (\mu_1^j, \ldots, \mu_r^j)$, where μ_i^j denotes the number of sign changes of u_j^* on (a_i, b_i), $i=1, \ldots, r$, with the condition that if $\mu_i^j \geqslant m_i$, then we set $\mu_i^j = m_i$. We will prove that $\boldsymbol{\mu}^{j+1} > \boldsymbol{\mu}^j$ where the ordering here is lexicographic, i.e., $(\mu_1^{j+1}, \ldots, \mu_r^{j+1}) > (\mu_1^j, \ldots, \mu_r^j)$ if and only if $\mu_i^{j+1} = \mu_i^j$, $i=1, \ldots, s-1$, and $\mu_s^{j+1} > \mu_s^j$, some $s=1, \ldots, r$. At each step j we have a u_j^*, as previously indicated, and a u^j which is constructed as was u^1 with respect to u_1^* for which $U(u^j) \subseteq U(u^1)$. We will show that if $\mu_i^j = m_i$, then $u^j(x)$ vanishes identically on (a_i, b_i). Since $u^j \in U(u^1)$, $u^j \not\equiv 0$, and there are only r intervals (a_i, b_i), $r \leqslant 2n$, this process cannot continue indefinitely, i.e., we must, after a finite number of steps, revert to Case II(i).

This is the idea behind the algorithm. We now prove these various claims. The construction of u^2 and u_2^* has been given. The general case is slightly different than this case. So let us prove directly our conclusion regarding u_2^* and u^2.

LEMMA 5.4. *$u^2(x)$ vanishes identically on (a_1, b_1) and $\boldsymbol{\mu}^2 > \boldsymbol{\mu}^1$.*

Proof. Since $u^2 \in U(u^1)$ and $U(u^1)|_{(a_1,b_1)}$ is a T-system of dimension m_1, either $u^2(x)$ vanishes identically on (a_1, b_1) or has at most $m_1 - 1$ zeros on (a_1, b_1). Assume the latter. u_2^* was constructed with at least m_1 sign changes on (a_1, b_1). By construction u^2 must have at least m_1 sign changes on (a_1, b_1). This contradicts the fact that u^2 has at most $m_1 - 1$ zeros on (a_1, b_1). Now, by construction, $u^1(x) \not\equiv 0$ on (a_1, b_1) and therefore we have $\mu_1^1 < m_1 = \mu_1^2$. Thus $\boldsymbol{\mu}^2 > \boldsymbol{\mu}^1$. ∎

This same method of proof shows

LEMMA 5.5. *If $\mu_i^j = m_i$, then $u_j(x)$ must vanish identically on (a_i, b_i).*

In general we construct u_{j+1}^* and u^{j+1} from u_j^* and u^j. However, note that we are always looking back to $U(u^1)$ rather than to $U(u^j)$.

The general construction is as follows: Given u_j^*, u^j, and $\boldsymbol{\mu}^j$, let k be the smallest index for which $u^j(x)$ does not vanish identically on (a_k, b_k). Recall that $u^j \in U(u^1)\backslash\{0\}$. Since $U(u^1)|_{(a_k,b_k)}$ is a T-system, and $U(u^j) \subseteq U(u^1)$, it follows from the results of Section 4 that $U(u^j)$ is a T-system on (a_k, b_k) of dimension $m_k^j \leqslant m_k$. By construction u^j has at least as many sign changes on (a_k, b_k) as u_j^*. Thus $m_k^j - 1 \geqslant \mu_k^j$. Since $u_j^*(x) = u^*(x)$ off M, it follows as in Case I that $u_j^* \notin U(u^j)$ on any subinterval of M. Thus there exists a $u_{j+1}^* = u_j^* - u$, $u \in U(u^j)$, such that u_{j+1}^* has at least m_k^j sign changes on (a_k, b_k). Furthermore, $u_{j+1}^*(x) = u_j^*(x)$ for all $x \in \bigcup_{i=1}^{k-1}(a_i, b_i)$ since $u(x) = 0$ on $\bigcup_{i=1}^{k-1}(a_i, b_i)$ for all $u \in U(u^j)$. Thus $\mu_i^{j+1} = \mu_i^j$, $i = 1,..., k-1$, and $\mu_k^{j+1} \geqslant m_k^j > \mu_k^j$. Hence $\boldsymbol{\mu}^{j+1} > \boldsymbol{\mu}^j$. To complete the construction we construct u^{j+1} with respect to u_{j+1}^* as u^1 was constructed with respect to u_1^*. ∎

Remark. If no $u \in U_n\backslash\{0\}$ vanishes on a set of positive measure, then the proof of Theorem 4.7 is immediate based on the first part of the proof.

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